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SOME BASIC LAWS OF ISOTROPIC TURBULENT FLOW

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SUMMARY

An investigation is made of the diffusion of artifi-
cially produced turbulence behind screens or other turbu-
lence producers. After defining the fundamental assumptions
underlying the statistical turbulence theory more accurately,
the author proposes a method that permits investigation of
the diffusion of turbulent disturbances of finite scale in
place of the "point source" disturbances considered by Von
Kármán. The method is based on the author's concept of
"disturbance moment" as a certain theoretically well-founded
measure of turbulent disturbances. Incidentally, with the
object of familiarizing the reader with the fundamentals of
the new theory, the author gives a presentation of the funda-
mentals of the theory in a form that is considered somewhat
simpler than that given in existing papers.

INTRODUCTION

In recent years, due chiefly to the investigations of
Taylor (reference 1) and Von Kármán (reference 2), further
progress has been made in the turbulence theory. The new
ideas are based, essentially, on the statistical turbulence
theory presented several decades ago by the great Soviet
physicist Friedmann, whose premature death occurred in 1925,
and by Keller (reference 3).

The principal difference between the new method and
previous statiscal methods lies in the introduction, to-
gether with the usual correlation moments of the velocity
at a given point of the flow, of special "association mo-
ments" (Friedmann's terminology) between the velocity com-
ponents at two different points of the flow at corresponding
instants of time. This idea has proved itself very fruitful

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and has enabled Taylor to develop the theory of the dissipation of turbulent disturbances behind screens and enabled Von Kármán to give a general equation for turbulence propagation.

The concept of association moment makes it possible to determine such an important magnitude as the "scale of the turbulence," a characteristic of turbulence of equal importance with the other characteristics; namely, the amplitudes and frequencies of the fluctuations. Modern physical experiments, such as are conducted in aerodynamics, permit measuring the association moments and thus check the results of the theory. From the practical point of view new theories are important because they replace semiempirical theories, they throw light on the physical structure of turbulent motion, and also because they provide a firm basis for the investigation of artificial flows in aerodynamic wind tunnels, open channels, and so forth. A knowledge of the turbulence characteristics of these flows permits a more accurate estimate of the scale effects of the phenomena and better application of the model tests to full-scale design.

A further development of the problem of the diffusion of artificially produced turbulence behind screens or other turbulence-producing devices is presented in this paper. With the fundamental assumptions of the theory more accurately defined a method is proposed that permits the investigation of the diffusion of turbulent disturbances of finite scale in place of the point source disturbances considered by Von Karman. The method is based on the author's concept of "moment of disturbance" as a certain theoretically well-founded measure of turbulent disturbances. Incidentally, with the object of familiarizing the reader with the fundamentals of the new theory, the latter is presented in a form that is considered simpler than that given in existing literature.

1. HOMOGENEOUS AND ISOTROPIC TURBULENCE — ASSOCIATION

MOMENTS AND CORRELATION TENSOR — TWO FIRST CORRELATION

FUNCTIONS FOR MOMENTS OF THE SECOND ORDER

Visualize a homogeneous turbulent flow, that is, a flow at a given instant the average characteristics of which are the same at different points. Take any two points in the flow M and M' and denote their relative radius vector $\overline{MM'}$ by r and its projections on the coordinate axes by ξ_1, ξ_2, ξ_3. The flow velocities at points M and M'' at the same instant

of time t are denoted by v and v' and their projections on the axes by v_i and v_i'. The average magnitudes:

$$\Phi_{ij} = \overline{v_i v_j'} \tag{1,1}$$

are called, following Friedmann and Keller, "association moments of the second order." These nine magnitudes constitute a tensor of the second rank Φ, called the tensor of the association moment. With regard to the averaging the usual averaging with respect to time of the turbulence theory is assumed. Use is also made of the generally assumed averaging quantities:

$$(1) \quad \overline{\varphi + \psi} = \overline{\varphi} + \overline{\psi}$$

$$(2) \quad \overline{\frac{\partial \varphi}{\partial s}} = \frac{\partial \overline{\varphi}}{\partial s}$$

$$(3) \quad \overline{\overline{\varphi}} = \overline{\varphi}$$

$$(4) \quad \overline{\overline{\varphi} \psi} = \overline{\varphi} \overline{\psi}$$

Equations (3) and (4) are applicable, as is known, for the condition that the average functions may be considered as constant (or slightly varying) functions in the averaging interval.

The tensor Φ in the general case of a homogeneous flow depends only on the relative position of the points M and M', that is, on the vector r and on the time. A particular case is where the association moment tensor is a function of the time and of the distance r between the points M and M' but not of their relative position in space. The turbulent flow in which the association moment tensor at a given instant does not depend on the direction in space of the line connecting the two points, but only on the distance between them, is called an isotropic turbulent flow. It should be particularly noted that the property of isotropy refers to the tensor as a whole as a physical magnitude and not to its individual components, which depend on the relative position of the line MM' and the coordinate axes.

The general form of the association moment tensor in a homogeneous, isotropic flow is established by means of the synthetic (physical) definition of the tensor. By the

definition of isotropy the components of the tensor Φ in the system of coordinates Mlnb, associated with the point M' (axis Ml is directed along $\overrightarrow{MM'}$, the axes Mn and Mb perpendicular to it), do not depend on the direction of these axes in space but only on the time and the distance r between the points M and M'. These components are denoted by:

(1) The association moment between the longitudinal components of the velocity along the vector MM' (with unit vector $\underline{l} = \frac{\underline{r}}{r}$) by

$$F(r, t) = \overline{v_l v_l'} \tag{1,2}$$

(2) The association moment between the velocity components transverse to MM' by

$$G(r, t) = \overline{v_n v_n'} = \overline{v_b v_b'} \tag{1,3}$$

(3) The association moments between one longitudinal and one transverse velocity component by

$$S(r, t) = \overline{v_l v_n'} = \overline{v_l v_b'}$$

$$S_1(r, t) = \overline{v_n v_l'} = \overline{v_b v_l'}$$

These tensor-defining magnitudes may be readily expressed in terms of the Cartesian components of the tensor $\Phi_{ij} = \overline{v_i v_j'}$.

$$\left. \begin{array}{l} F(r, t) = \overline{v_l v_l'} = \overline{(\underline{v} \cdot \underline{l})(\underline{v}' \cdot \underline{l})} = \overline{v_i l_i v_j' l_j} = \Phi_{ij} l_i l_j \\[2mm] G(r, t) = \overline{v_n v_n'} = \overline{(\underline{v} \cdot \underline{n})(\underline{v}' \cdot \underline{n})} = \overline{v_i n_i v_j' n_j} = \Phi_{ij} n_i n_j \\[2mm] S(r, t) = \Phi_{ij} l_i n_j, \quad S_1(r, t) = \Phi_{ij} n_i l_j \end{array} \right\} \tag{1,4}$$

where $\underline{v} \cdot \underline{l}$ denotes the scalar product; l_i, n_j are the projections of the unit vectors $\underline{l}$ and $\underline{n}$, that is, the direction cosines of the vector MM' and any arbitrary vector perpendicular to it. The summation sign is omitted for repeated indices.

By the definition of isotropy the moments F, G, S, and S_1 should not depend on the direction of MM' and the axes perpendicular to it, that is, on the magnitudes l_i and n_j. The dependence of the tensor Φ on the unit vector $\underline{l}$ and the scalars r and t should be of the general form

$$\Phi = A(r,t)\, \underline{ll} + B(r,t)\, I \qquad (1,5)$$

where $\underline{A}(r,t)$ and $B(r,t)$ are as yet undetermined functions, $\underline{ll}$ is the symbol for the dyad constructed from the vector $\underline{l}$, and I is the unit tensor. Analytically the result is:

$$\Phi_{ij} = A(r,t)\, l_i l_j + B(r,t)\, I_{ij}, \quad I_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \qquad (1,5')$$

Hence, substituting from (1,5') in (1,4) while making use of the known properties of direction cosines gives:

$$F(r,t) = \underline{A}(r,t)\, l_i l_j l_i l_j + B(r,t)\, I_{ij} l_i l_j = \underline{A} + B$$

$$G(r,t) = \underline{A}(r,t)\, l_i l_j n_i n_j + B(r,t)\, I_{ij} n_i n_j = B$$

$$S(r,t) = A(r,t)\, l_i l_j l_i n_j + B(r,t)\, I_{ij} l_i n_j = 0$$

$$S_1(r,t) = A(r,t)\, l_i l_j n_i l_j + B(r,t)\, I_{ij} n_i l_j = 0$$

The equating to zero of the components $\overline{v_l v_n'}$, $\overline{v_l v_b'}$ and other nondiagonal components shows that the axes Mlnb are the principal axes of the tensor Φ. Consequently

$$A = F - G, \qquad B = G$$

and the final expression for the tensor φ will be

$$\left. \begin{array}{l} \Phi = (F - G)\ \underline{11} + GI \\[2mm] \Phi_{ij} = (F - G)\ l_i l_j + GI_{ij} \end{array} \right\} \qquad (1,6)$$

In what follows the functions $F(r,t)$ and $G(r,t)$ are denoted as the first and second moment functions.[*]

Noting that in the case of homogeneous isotropic turbulence

$$\overline{v_1^2} = \overline{v_2^2} = \overline{v_3^2} = \overline{v_1'^2} = \overline{v_2'^2} = \overline{v_3'^2} = \overline{v^2}$$

Von Kármán, in place of the association moment tensor Φ, introduces the correlation tensor

$$R = \frac{1}{\overline{v^2}}\ \Phi, \qquad R_{ij} = \frac{\overline{v_i v_j'}}{\sqrt{\overline{v^2}\overline{v'^2}}} = \frac{\overline{v_i v_j'}}{\overline{v^2}} \qquad (1,7)$$

Together with Von Kármán the two correlation functions are introduced:

$$f(r,t) = \frac{F(r,t)}{\overline{v^2}} \qquad g(r,t) = \frac{G(r,t)}{\overline{v^2}} \qquad (1,8)$$

so that, making use of the evident relation $l_i = \xi_i / r$

$$\left. \begin{array}{l} R = \dfrac{f - g}{r^2}\ \underline{rr} + gI \\[4mm] R_{ij} = \dfrac{f - g}{r^2}\ \xi_i \xi_j + g^r{}_{ij} \end{array} \right\} \qquad (1,9)$$

or

Incidentally it should be noted that $\overline{v_i v_j} = 0$ for $i \neq j$ as r approaches zero in the expressions for S or S_1. This is not surprising since the average velocities

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[*]The reader is reminded that these functions have a simple physical meaning; namely, the association moments between two longitudinal or two transverse velocity components with respect to the line connecting the points.

in a homogeneous turbulence are everywhere the same and there is no momentum transport. The friction is then also evidently equal to zero.

The unknown correlation functions f and g are defined by a relation based on the equation of continuity.

2. ASSOCIATION MOMENTS OF THE THIRD ORDER.— ASSOCIATION

MOMENT TENSOR OF THE THIRD RANK AND ITS MOMENT FUNCTIONS.—

CORRELATION TENSOR AND ITS CORRECTION FUNCTIONS

In the Friedmann and Keller theory the association moments of the third order were not considered. It was assumed that these moments were small and that they could be neglected. Von Kármán at first neglected them and only after the criticism of Taylor did he give up this simplification. Since the investigation of the problem of isotropic turbulent motion is to be made with a minimum of quantitative restricting assumptions the theory of association moments of the third order is presented by the same method of presentation as in the previous section.

By association moments of the third order or simply third moments are meant average magnitudes of the type:

$$\overline{v_i v_j v_k'} \quad \text{and} \quad \overline{v_i v_j' v_k'}$$

where the notation remains the same.

It is readily seen that the association moments containing two components at point M' and one at point M may be expressed in terms of the association moments containing two components at point M and one at point M'. Interchanging the places of the points M and M' is equivalent to reversing the direction of the vector $\overline{MM'}$ so that

$$\overline{v_i v_j' v_k'} = -\overline{v_i' v_j v_k} \qquad \text{and so forth}$$

Therefore, consider the magnitudes of the type $\overline{v_i v_j v_k'}$. The set of these magnitudes form an association moment tensor

of the third rank. Denoting this tensor by Π and its
components by Π_{ijk} gives its moment functions by consid-
ering, as previously, the synthetic (physical) definition
of the tensor. For this purpose consider the components
of the tensor Π on the axes which are the principal axes
of the tensor Φ:

$$\overline{v_l^2 v_l{}'}, \quad \overline{v_l v_n v_l{}'}, \quad \overline{v_l v_n v_n{}'}, \quad \overline{v_n^2 v_l{}'}, \quad \overline{v_n^2 v_n{}'} \quad \text{and so forth}$$

As in the previous case it is noted that (repeated subscripts
denote summation):

$$v_l = v_1 l_1, \quad v_n = v_1 n_1, \quad v_l{}' = v_1{}' l_1 \quad \text{and so forth}$$

Hence

$$\overline{v_l^2 v_l{}'} = \overline{v_1 l_1 v_j l_j v_k{}' l_k} = \Pi_{ijk} l_1 l_j l_k \left.\vphantom{\begin{array}{c} \\ \\ \\ \\ \\ \end{array}}\right\}$$

$$\overline{v_l v_n v_l{}'} = \Pi_{ijk} l_1 n_j l_k \qquad \qquad (2,1)$$

$$\overline{v_l v_n v_n{}'} = \Pi_{ijk} l_1 n_j n_k \quad \text{and so forth}$$

According to the condition of isotropy all these third
moments should not depend on the direction cosines l_i and
n_j. The equation of the general dependence of the tensor
of the third rank Π on the vector l (invariant equal to
unity) and the scalars r and t reads:

$$\Pi_{ijk} = A l_i l_j l_k + B l_i I_{jk} + C l_j I_{ik} + D l_k I_{ij} \quad (2,2)$$

where A, B, C, and D are scalar functions of r and t.

To determine these scalar functions the values Π_{ijk}
from (2,2) are substituted in (2,1) so that, for example:

$$\overline{v_l^2 v_l{}'} = A l_i l_1 l_j l_j l_k l_k + B l_i l_1 I_{jk} l_j l_k + C l_j l_j I_{ik} l_1 l_k + D l_k l_k I_{ij} l_1 l_j$$

or, by making use of the known properties of the direction cosines:

$$l_1 l_1 = 1, \quad l_{jk} l_j l_k = l_j l_j = 1, \text{ and, so forth}$$

$$\overline{v_1{}^2 v_1{}'} = A + B + C + D$$

In a similar manner

$$\overline{v_1 v_n v_1{}'} = 0, \quad \overline{v_1 v_n v_n{}'} = B = \overline{v_n v_1 v_n{}'} = C$$

$$\overline{v_n{}^2 v_1{}'} = D, \quad \overline{v_n{}^2 v_n{}'} = 0, \quad \overline{v_1{}^2 v_n{}'} = 0 \quad \text{etc.}$$

the three nonzero moment functions are denoted thus:

$$\overline{v_1{}^2 v_1{}'} = K \ (r,t); \quad \overline{v_1 v_n v_n{}'} = \overline{v_1 v_b v_b{}'} = Q \ (r,t);$$

$$\overline{v_n{}^2 v_1{}'} = \overline{v_b{}^2 v_1{}'} = H \ (r,t) \tag{2,3}$$

For the coefficients A, B, C, and D it results in:

$$A = K - 2Q - H$$
$$B = C = Q$$
$$D = H$$

The final expression for the components of the association moment tensor of the third rank is:

$$\Pi_{1jk} = (K - 2Q - H) l_1 l_j l_k + Q l_1 I_{jk} + Q l_j I_{1k} + H l_k I_{1j} \tag{2,4}$$

In place of the tensor of association moments of the third order Von Karman uses the correlation tensor:

$$T = \frac{1}{(\overline{v^2})^{\frac{3}{2}}} \Pi; \quad T_{ijk} = \frac{\Pi_{ijk}}{(\overline{v^2})^{\frac{3}{2}}} = \frac{\overline{v_i v_j v_k{}'}}{(\overline{v^2})^{\frac{3}{2}}} \qquad (2,5)$$

If the correlation functions corresponding to the moment functions are denoted by the corresponding small letters, setting

$$k(r,t) = \frac{K(r,t)}{(\overline{v^2})^{\frac{3}{2}}}; \quad q(r,t) = \frac{Q(r,t)}{(\overline{v^2})^{\frac{3}{2}}}; \quad h(r,t) = \frac{H(r,t)}{(\overline{v^2})^{\frac{3}{2}}} \quad (2,6)$$

and l_i is replaced by ξ_i/r, the Von Kármán formula reads:

$$T_{ijk} = \frac{k-h-2q}{r^3} \xi_i \xi_j \xi_k + \frac{h}{r} I_{ij}\xi_k + \frac{q}{r} I_{ik}\xi_j + \frac{q}{r} I_{jk}\xi_i \qquad (2,7)$$

4. VECTOR OF ASSOCIATION MOMENT RELATING PRESSURE AND

VELOCITY IN ISOTROPIC FLOW AND CORRESPONDING

MOMENT FUNCTION – DERIVATION OF FUNDAMENTAL

RELATIONS BETWEEN MOMENT FUNCTIONS

The investigation of the problems of the dynamics of an isotropic turbulent flow requires still another association moment relating the pressure p at a point M with the velocity vector v' at a neighboring point. This association moment is represented by the vector $\overline{pv'}$ with the projections $\overline{pv_i{}'}$. Following the method of the preceding section the physical components of the vector of the association moment relating the pressure with the longitudinal and transverse velocities are expressed by the projections:

$$\overline{pv_l{}^r} = \overline{pv_i{}'l_i}; \quad \overline{pv_n} = \overline{pv_i n_i};$$
$$\overline{pv_b} = \overline{pv_i b_i} \qquad (4,1)$$

on the condition that, because of the isotropy, these physi-
cal magnitudes do not depend on l_1, n_1, and b_1. The con-
dition of the general isotropic dependence of the vector $\overline{pv}$
on the unit vector l and the scalars r and t is evident-
ly reduced to the following:

$$\overline{pv_1'} = P(r,t)\, l_1 \tag{4,2}$$

where $P(r,t)$ is an arbitrary scalar function of the varia-
bles r,t. The physical meaning of these functions will
become clear if the value $\overline{pv_1'}$ is substituted from (4,2)
in (4,1); it is:

$$\left.\begin{array}{l}\overline{pv_1'} = P(r,t)\, l_1 l_1 = P(r,t) \\[6pt] \overline{pv_n'} = P(r,t)\, l_1 n_1 = 0 \\[6pt] \overline{pv_b'} = P(r,t)\, l_1 b_1 = 0 \end{array}\right\} \tag{4,3}$$

Hence the moment function $P(r,t)$ represents the association
moment relating the pressure at point M with the longitudi-
nal velocity component at point $M^\dagger$; the association moment
of the pressure with the transverse velocity component is
equal to zero. Elsewhere it is shown that $P(r,t)$ must also
be equal to zero by reason of the continuity equation.

The foregoing statement regarding the association moment
of the pressure and velocity naturally remains true for the
association moments of any scalar function and the velocities.
This remark will be of use in the following information.

Simple transformations on the continuity equation,
which, of course, holds for the turbulent flow, enables cer-
tain relations between the moment functions of the associa-
tion moments to be found:

$$P(r,t);\quad F(r,t);\quad G(r,t);\quad K(r,t);\quad Q(r,t);\quad H(r,t)$$

Proceeding from the simplest and keeping the point M
fixed, write the equation of continuity in the neighborhood
of the point M' (repeated indices denote summation)

$$\frac{\partial v_j'}{\partial \xi_j} = 0 \qquad\qquad (4,4)$$

Multiply both sides of the above equation by the pressure p, which does not depend on ξ at point M, and take the average; this yields

$$\frac{\partial \overline{p v_j'}}{\partial \xi_j} = 0 \qquad\qquad (4,5)$$

But, by (4,2)

$$\overline{p v_j'} = P(r,t) l_j = P(r,t) \frac{\xi_j}{r} = \pi(r,t) \xi_j$$

Therefore, (4,4) may be written thus:

$$3\pi + \xi_j \frac{\partial \pi}{\partial \xi_j} = 3\pi + \xi_j \pi' \frac{\xi_j}{r} = 3\pi + r\pi' = 0$$

where the prime denotes differentiation with respect to r. The solution of this equation is

$$\pi = \frac{C(t)}{r^3}; \quad P = \frac{C(t)}{r^2}$$

From the condition of finiteness of P for $r \longrightarrow 0$ follows

$$C(t) = 0, \quad P(r,t) = 0$$

Thus in a homogeneous isotropic turbulent flow the association moment of the pressure at one point with the velocity at another is equal to zero, that is,

$$\overline{p v_l'} = \overline{p v_n'} = \overline{p v_b'} = 0$$

or

$$\overline{p \underline{v}'} = 0$$

Multiplying both sides of equation (4,4) by v_1, independent of ξ_1, and averaging gives

$$\frac{\partial \overline{v_1 v'}_j}{\partial \xi_j} = 0 \tag{4,6}$$

The substitution of the expressions for the components of the tensor in terms of the moment functions F and G by (1,6) first replacing l_1 by ξ_1/r gives

$$\frac{\partial}{\partial \xi_j}\left(\frac{F-G}{r^2}\,\xi_1 \xi_j\right) + \frac{\partial}{\partial \xi_j}(G I_{1j}) = 0$$

or

$$\frac{F-G}{r^2}\frac{\partial}{\partial \xi_j}(\xi_1 \xi_j) + \xi_1 \xi_j \frac{\partial}{\partial \xi_j}\left(\frac{F-G}{r^2}\right) + \frac{\partial G}{\partial \xi_1} = 0$$

Noting that (prime denotes partial differentiation with respect to r):

$$\frac{\partial}{\partial \xi_j}(\xi_1 \xi_j) = \xi_1 \frac{\partial \xi_1}{\partial \xi_j} + \xi_j \frac{\partial \xi_1}{\partial \xi_j} = 3\xi_1 + \xi_1 = 4\xi_1$$

$$\frac{\partial}{\partial \xi_1}\frac{F-G}{r^2} = \left(\frac{F-G}{r^2}\right)' \frac{\xi_1}{r}; \quad \frac{\partial G}{\partial \xi_1} = G'\,\frac{\xi_1}{r}$$

leaves after simple reductions:

$$F' + 2\,\frac{F-G}{r} = 0 \tag{4,7}$$

This is the relation between the moment functions F and G. With the aid of this relation G is replaced by F according to the equation:

$$G = F + \frac{1}{2}\,rF' \tag{4,8}$$

Next, consider the moments of the third order. In the
same manner as before, multiplying both sides of equation
(4,4), where the index of summation j is replaced by k,
by $v_i v_j$ and averaging the results in:

$$\frac{\partial \overline{v_i v_j v'_k}}{\partial \xi_k} = 0$$

Substitution in the above equation of the values of the
components of Π_{ijk} by (2,4) gives:

$$\frac{\partial}{\partial \xi_k}\left(\frac{K-2Q-H}{r^5}\,\xi_i \xi_j \xi_k\right) + I_{jk}\,\frac{\partial}{\partial \xi_k}\left(\frac{Q}{r}\,\xi_i\right)$$

$$+ I_{ik}\,\frac{\partial}{\partial \xi_k}\left(\frac{Q}{r}\,\xi_j\right) + I_{ij}\,\frac{\partial}{\partial \xi_k}\left(\frac{H}{r}\,\xi_k\right) = 0 \qquad (4,9)$$

or

$$\frac{K-2Q-H}{r^3}\,\frac{\partial}{\partial \xi_k}\,(\xi_i \xi_j \xi_k) + \xi_i \xi_j \xi_k \left(\frac{K-2Q-H}{r^3}\right)'\,\frac{\xi_k}{r}$$

$$+ \frac{\partial}{\partial \xi_j}\left(\frac{Q}{r}\,\xi_i\right) + \frac{\partial}{\partial \xi_i}\left(\frac{Q}{r}\,\xi_j\right) + I_{ij}\,\frac{\partial}{\partial \xi_k}\left(\frac{H}{r}\,\xi_k\right) = 0 \qquad (4,10)$$

But, as may readily be seen .

$$\frac{\partial}{\partial \xi_k}\,(\xi_i \xi_j \xi_k) = \xi_k\,\frac{\partial}{\partial \xi_k}\,(\xi_i \xi_j) + \xi_i \xi_j\,\frac{\partial \xi_k}{\partial \xi_k} = 2\xi_i \xi_j + 3\xi_i \xi_j = 5\xi_i \xi_j$$

$$\frac{\partial}{\partial \xi_j}\left(\frac{Q}{r}\,\xi_i\right) = \xi_i \left(\frac{Q}{r}\right)'\,\frac{\xi_j}{r} + \frac{Q}{r}\,\frac{\partial \xi_i}{\partial \xi_j} = \frac{1}{r}\left(\frac{Q}{r}\right)'\,\xi_i \xi_j + I_{ij}\,\frac{Q}{r}$$

$$\frac{\partial}{\partial \xi_i}\left(\frac{Q}{r}\,\xi_j\right) = \frac{1}{r}\left(\frac{Q}{r}\right)'\,\xi_i \xi_j + I_{ij}\,\frac{Q}{r}$$

$$\frac{\partial}{\partial \xi_k}\left(\frac{H}{r}\,\xi_k\right) = \frac{H}{r}\,\frac{\partial \xi_k}{\partial \xi_k} + \xi_k \left(\frac{H}{r}\right)'\,\frac{\xi_k}{r} = 3\,\frac{H}{r} + r\left(\frac{H}{r}\right)'$$

Hence equation (4, 10) gives

$$\left[5\, \frac{K - 2Q - H}{r^3} + r\left(\frac{K - 2Q - H}{r^3} \right)' + \frac{2}{r} \left(\frac{Q}{r} \right)' \right] \xi_i \xi_j$$

$$+ \left[2\, \frac{Q}{r} + 3\, \frac{H}{r} + r\left(\frac{H}{r} \right)' \right] I_{ij} = 0 \dots$$

In view of the fact that this equation must be satisfied for any ξ_i, the expressions in brackets are individually equaled to zero; hence after simple reductions, the two equations connecting the moment functions K, Q, and H read as follows:

$$\left. \begin{array}{c} K' - H' + \dfrac{2K - 2H - 6Q}{r} = 0 \\[3mm] H' + \dfrac{2Q + 2H}{r} = 0 \end{array} \right\} \qquad (4,11)$$

The last equation expresses Q in terms of H:

$$Q = - H - \frac{r}{2}\, H' \qquad\qquad (4,12)$$

Substitution of this value of Q in the first equation (4,11) gives:

$$(K + 2H)' + \frac{2}{r}\,(K + 2H) = 0$$

hence

$$K + 2H = \frac{C(t)}{r^3}$$

From the condition of finiteness of the sum $K + 2H$ for $r \rightarrow 0$ follows:

$$K + 2H = 0 \qquad\qquad (4,13)$$

It is readily seen that equation (4,13) is satisfied because of the previously proved general property of isotropic flow;

namely, that the association moments of the scalars at one point with the velocity components at another are equal to zero. Thus:

$$K + 2H = \overline{v_l{}^2 v_l{}'} + 2\overline{v_n{}^2 v_l{}'} = \overline{v_l{}^2 v_l{}'} + \overline{v_n{}^2 v_l{}'} + \overline{v_b{}^2 v_l{}'} = \overline{v^2 v_l{}'} = 0$$

This equation may be considered as equivalent to

$$\overline{\frac{\rho v^2}{2} \underline{v}'} = 0 \qquad ; \qquad (4,14)$$

where $\rho v^2/2$ is the kinetic energy at point M, and $\underline{v}'$ the velocity at point M'. Bearing in mind the statement concerning the association moment of the pressure and velocity, write:

$$\overline{\left(p + \frac{\rho v^2}{2}\right) \underline{v}'} = 0 \qquad (4,15)$$

Thus in a homogeneous isotropic turbulent flow there is no correlation between the velocity at one point and the total mechanical energy at another.

The same holds true for magnitudes measured at the same point in the limit as $r \rightarrow 0$. According to the foregoing the isotropy of a turbulent flow is intimately related with the absence of momentum transport (friction). Now it is found that in an isotropic flow there can be no transport of energy.

It should be borne in mind that the obtained results are not a trivial consequence of the purely kinematic symmetry of isotropic flow (this is true, for example, with regard to the equations:

$$\overline{p v_n{}'} = 0; \qquad \overline{v^2 v_n{}'} = 0$$

and so forth, which might have been derived by rotating n through 180^0). They were proved by the equation of continuity of the flow (the equations $\overline{p v_l{}'} = 0$ and $\overline{v^2 v_l{}'} = 0$ cannot be derived from considerations of symmetry). This equation is applicable only to incompressible fluid.

5.   FUNDAMENTAL EQUATION OF THE PROPAGATION OF TURBULENCE

IN AN ISOTROPIC FLOW — DISTURBANCE MOMENT — THEOREM ON THE

CONSERVATION OF THE DISTURBANCE MOMENT — ANALOGY BETWEEN

THE PROPAGATION OF TURBULENT DISTURBANCES AND

THE PROPAGATION OF HEAT

The next step is the derivation of the fundamental dif-
ferential equation of the propagation of turbulent disturb-
ances as recently given by Von Kármán (1938). With a view
to further investigation this equation is derived in terms of
moment functions and not the correlations by Von Kármán.

The continuity equation, as a homogeneous equation with
respect to the velocities, affords expressions of the moment
functions only in terms of others of the same order. A re-
lation between the moment functions of different orders is
obtained by the Navier-Stokes equations, which are written
down for the point M.

$$\frac{\partial v_1}{\partial t} + v_j \frac{\partial v_1}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_1} + \nu \nabla_x^2 v_1 ; \left(\nabla_x^2 = \frac{\partial}{\partial x_1^2} + \frac{\partial}{\partial x_2^2} + \frac{\partial}{\partial x_3^2} \right)$$

Multiplying both sides by v_k', that is, by the velocity
at point M', and averaging gives

$$\overline{v_k' \frac{\partial v_1}{\partial t}} + \overline{v_k' v_j \frac{\partial v_1}{\partial x_j}} = -\frac{1}{\rho}\overline{v_k' \frac{\partial p}{\partial x_1}} + \nu \overline{v_k' \nabla_x^2 v_1} \tag{5,1}$$

The component v_k' is a function of the time and the coordi-
nates ξ_i; and not of x_i; then by use of the equation of
continuity the triple product on the left side can be written
as:

$$\overline{v_k' v_j \frac{\partial v_1}{\partial x_j}} = \overline{v_k' \frac{\partial x_1 v_j}{\partial x_f}} - \overline{v_k' v_1 \frac{\partial v_j}{\partial x_1}} = \frac{\partial \overline{v_1 v_j v_k'}}{\partial x_j}$$

The association moment $\overline{v_1 v_j v_k'}$ may be considered as a
function of x_j if the point M' is held fixed or as a

function of ξ_j if the point M is held fixed, differenti-
ation with respect to x_j corresponding to fixing the point
M' and differentiation with respect to ξ_j to fixing the
point M, that is, to a reversal in the direction of differ-
entiation; hence

$$\frac{\partial \overline{v_i v_j v_k'}}{\partial x_j} = -\frac{\partial \overline{v_i v_j v_k'}}{\partial \xi_j}$$

so that

$$\overline{v_k' v_j \frac{\partial v_i}{\partial x_j}} = -\frac{\partial}{\partial \xi_j} \overline{v_i v_j v_k'} \qquad (5,2)$$

From the foregoing further follows:

$$\overline{v_k' \frac{\partial p}{\partial x_i}} = \frac{\partial}{\partial x_i} \overline{v_k' p} \equiv 0 \qquad (5,3)$$

and

$$\overline{v_k' \nabla_x^2 v_i} = \nabla_x^2 \overline{v_i v_k'} = \nabla^2 \overline{v_i v_k'} \qquad (5,4)$$

where the symbol ∇^2 denotes the Laplacian with respect to
the variable ξ (the sign before the Laplacian evidently
should not vary). After the above transformations equation
(5,1) becomes

$$\overline{v_k' \frac{\partial v_i}{\partial t}} - \frac{\partial}{\partial \xi_j} \overline{v_i v_j v_k'} = \nu \nabla^2 \overline{v_i v_k'} \qquad (5,5)$$

The same process is repeated by interchanging the points M
and M'; that is, write down the equation of Navier—Stokes
at the point M', multiply both sides by v_i, and so forth,
so that instead of equation (5,5) there is:

$$\overline{v_i \frac{\partial v_k'}{\partial t}} + \frac{\partial}{\partial \xi_j} \overline{v_i v_j' v_k'} = \nu \nabla^2 \overline{v_i v_k'} \qquad (5,6)$$

But as already noted in section 2:

$$\overline{v_i v_j' v_k} = -\overline{v_i' v_j v_k}$$

hence (5,6) changes to

$$\overline{v_i \frac{\partial v_k'}{\partial t}} - \frac{\partial}{\partial \xi_j} \overline{v_j v_k v_i'} = \nu \nabla^2 \overline{v_i v_k'} \qquad (5,7)$$

Combining (5,5) and (5,7) and recalling the previously assumed notation for the components of the tensors of the association moments finally gives:

$$\frac{\partial \Phi_{ik}}{\partial t} - \frac{\partial}{\partial \xi_j} \left(\Pi_{ijk} + \Pi_{jki} \right) = 2\nu \nabla^2 \Phi_{ik} \qquad (5,8)$$

The foregoing complicated system of equations can be very much simplified by expressing all the components of the tensors in terms of the corresponding moment functions. Making use of the condition that the equation must hold for any values of ξ_i as before affords, however, two equations with three unknown independent moment functions (for example, F, G, and H). Eliminating one of these (for example, G), after some reductions which are disregarded, gives the following equation:

$$\frac{\partial F}{\partial t} + 2 \left(\frac{\partial F}{\partial r} + \frac{4}{r} H \right) = 2\nu \left(\frac{\partial^2 F}{\partial r^2} + \frac{4}{r} \frac{\partial F}{\partial r} \right) \qquad (5,9)$$

In the foregoing the derivatives of H and F with respect to r were denoted by a prime, the time being considered as fixed throughout. Here the partial derivative notation is used in order to bring out the character of the time as an independent variable.

Equation (5,9) was given in 1938 by Von Kármán in terms of the correlation functions f and h in the form:

$$\frac{\partial(\overline{v}^2 f)}{\partial t} + 2(\overline{v}^2)^{\frac{3}{2}} \left(\frac{\partial h}{\partial r} + \frac{4}{r} h\right) = 2v\overline{v}^2 \left(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r} \frac{\partial f}{\partial r}\right) \quad (5,10)$$

The form (5,9) proposed here is more suitable for application as will be shown later since it does not contain the artificially introduced factor $\overline{v}^2$. The equation by Von Kármán represents a single equation with two unknown functions h and f, that is, an indeterminate equation. As has been shown by Von Karman, in his most recent paper, it is impossible to render the equation determinate by using the same method but passing to moments of a higher order because the number of the moment or correlation functions increases together with the number of new equations (this fact has long been known due to the investigations by Friedmann as far back as 1925).

Equation (5,9) may be considered as an equation determining the distribution in space and time of the association moment $F(r,t)$ or, what is equivalent, of a magnitude proportional to it; namely, the coefficient of correlation between two longitudinal velocities at two neighboring points at a distance r from each other at the same instant of time. The first term determines the local change of F, and the remaining terms on the left side give the convective change in F expressed in terms of the function H, and finally the right side gives the molecular diffusion of the same magnitude F. The indeterminateness of the equation is due to the presence of the convective terms which remain unknown and cannot be expressed in terms of the function F without any additional assumptions. Equation (5,9) might have been directly arrived at by averaging the Navier-Stokes equations in spherical coordinates, and it would then be clear that the local and diffusion terms represent simultaneously an averaging of the fluctuations $v_1 v_1{}'$ and a fluctuation of the average $\overline{v_1 v_1{}'}$ while the convective term, on account of nonlinearity, determines the mean convection of the magnitude $v_1 v_1{}'$ but not the convection of the mean $\overline{v_1 v_1{}'} = F$; and precisely herein lies the difficulty of the problem.

The Von Kármán theory in 1937 in which the effect of moments of the third order was omitted, that is, the con-vective (inertia) terms, may be regarded only as a theory of pure diffusion without convection, that is, a motion with very small Reynolds numbers. Attempts to consider the problem

of turbulent isotropic motion for large Reynolds numbers in
Von Kármán's recent paper (1938) are as yet in a very primi-
tive stage. In this connection it is of great interest to
clarify the general properties of isotropic turbulent flow
in the general case of motion with both small and large
Reynolds numbers.

Subsequently proof is given of a general theorem of
turbulent disturbances: namely, a theorem on the conserva-
tion of the disturbance moment. With the aid of this theorem
the problem of the decay of turbulent disturbances can be set-
up and solved.

6.   THEOREM ON THE CONSERVATION OF THE DISTURBANCE MOMENT —

DISTURBANCE MOMENT AS A MEASURE OF THE QUANTITY OF

DISTURBANCE — ANALOGY WITH HEAT

PROPAGATION PROBLEMS

Before proceeding to the derivation of the fundamental
theorem a few remarks concerning the physical significance of
the problem of obtaining the function $F(r,t)$ or $f(r,t) = \frac{1}{\overline{v^2}} F(r,t)$ should be of interest. Assume that the function
$F(r,t)$ is determined; then the correlation coefficient
$f(r,t) = \overline{v_1 v_1'}/\overline{v^2}$ between the longitudinal velocities will
be determined. For $r = 0$ the correlation coefficient is
evidently equal to unity and a complete relation between the
phenomena exists. With increasing r, however, the corre-
lation coefficient rapidly decreases corresponding to a
decrease of the statistical association between the phenomena
at the points M and M'. For $r = \infty$ F and f are evidently
equal to zero.

Consider the integral

$$L = \int_0^\infty f(r,t)dr = \frac{1}{\overline{v^2}} \int_0^\infty F(r,t)\, dr \qquad (6,1)$$

This integral may be visualized as a certain length, derived
with the aid of the correlation coefficient, that character-
izes the mean dimension of the region of disturbance or, as

termed later on, the scale of the turbulence. The magnitude $\sqrt{\overline{v^2}} = \sqrt{F(0,t)}$ is denoted as the intensity of the turbulence (it is the square root of the mean square of the velocity).

If, at a given instant, disturbances are produced in a stationary fluid (by passing a screen through it), then under the effect of viscosity and convection these disturbances will be propagated in space and dissipated on account of the viscosity. It can readily be seen that the intensity will decrease to zero and the scale of turbulence will expand as a result of the diffusion and convection. The question naturally arises whether or not a certain quantity will be conserved with respect to time. It will be shown that under very general assumptions such a magnitude that remains constant in time exists and may serve as a measure of the quantity of disturbance externally applied to the fluid. To prove this both sides of equation (5.9) are multiplied by r^k where k for the present is assumed to be positive and integrated with respect to r between the limits of zero and infinity. Then

$$\frac{\partial}{\partial t} \int_0^\infty F\, r^k\, dr + 2 \int_0^\infty \frac{\partial H}{\partial r} r^k dr + 8 \int_0^\infty Hr^{k-1}\, dr$$

$$= 2v \left(\int_0^\infty \frac{\partial^2 F}{\partial r^2} r^k dr + 4 \int_0^\infty \frac{\partial F}{\partial r} r^{k-1}\, dr \right) \qquad (6,2)$$

Integrating (formally) by parts gives

$$\frac{d}{dt} \int_0^\infty Fr^k dr = -2 \left(Hr^k\right)_0^\infty + 2k \int_0^\infty Hr^{k-1}\, dr - 8 \int_0^\infty Hr^{k-1}\, dr$$

$$+ 2v \left(\frac{\partial F}{\partial r} r^k\right)_0^\infty - 2vk \int_0^\infty \frac{\partial F}{\partial r} r^{k-1}\, dr + 8v \int_0^\infty \frac{\partial F}{\partial r} r^{k-1}\, dr$$

$$= -2 \left(Hr^k\right)_0^\infty + (2k-8)_0^\infty Hr^{k-1}\, dr + 2v \left(\frac{\partial F}{\partial r} r^k\right)_0^\infty$$

$$- 2v(k-4) \int_0^\infty \frac{\partial F}{\partial r} r^{k-1}\, dr \qquad (6,3)$$

As r increases to infinity the functions F, H, ∂f/∂r which are proportional to the correlation coefficients, must rapidly decrease to zero. For small r the function F, being an even function, has the form

$$F(r,t) = F(0,t) + F_0'' (0,t) \frac{r^2}{2} + \ldots = \overline{v}^2 + F_0'' \frac{r^2}{2} + \ldots$$

and, therefore, for small r:

$$\frac{\partial F}{\partial r} \approx r$$

Similarly the function H(r,t) is expanded in a series:

$$H(r,t) = H(0,t) + H'(0,t) \, r + \frac{1}{2} H''(0,t)r^2 + \frac{1}{6} H'''(0,t)r^3 + \ldots$$

It is noted that, on account of the isotropy, on reversing the direction of l to l₁ and passing to the limit as r → 0, the value of H(0,t) does not change, but

$$H(0,t) = \overline{v_n{}^2 v}\Big|_{r \to 0} = \overline{v_n{}^2 v}_{l_1}\Big|_{r \to 0}$$

and, therefore, H(0,t) = 0.

By following the same reasoning it is found that on account of the isotropy all coefficients of the series up to H''(0,t) inclusive, must become zero. Hence for small r

$$H(r,t) \approx r^3$$

Returning to equation (6.3) it is assumed that the rate of decrease of the functions H, F, and ∂F/∂r as r → ∞ exceeds the rate of increase of r^k as r → ∞ and k = 4. From (6.3) for k = 4 it follows

$$\frac{d}{dt} \int_0^\infty F(r,t)r^4 dr = 0 \qquad (6,4)$$

in other words for all values of t

$$M = \int_0^\infty F(r,t)r^4 dr = \text{constant} = \int_0^\infty F(r,0)r^4 dr = M_0 \quad (6,5)$$

The magnitude M which remains constant in time notwithstanding the diffusion of the disturbance is termed "disturbance moment" and serves as a measure for the turbulent disturbance. In the same way as in the phenomenen of heat diffusion where the total quantity of heat initially imparted to the fluid remains the same, the integral (6,5) represents a certain measure of the quantity of disturbance which remains the same notwithstanding the dissipation of the intensity of the turbulence in the flow.

It is emphasized that formula (6,5) was derived for the case of homogeneous, isotropic turbulence from equation (5,9) in its general form without rejecting the convective term, that is, from the indeterminate equation. Probably in the case of nonhomogeneous and nonisotropic turbulence there exists an analogy, as yet unknown, corresponding to formula (6,5).

Therefore, the following general theorem – the disturbance moment in a homogeneous, isotropic turbulent flow – remains constant and is determined by the initial disturbance imparted to the fluid. In what follows, this theorem is termed "the theorem of the conservation of the disturbance moment."

The foregoing equation of the conservation of the disturbance moment may be readily interpreted as follows; Consider, together with the earlier introduced scale of turbulence L determined by the integral (6,1), another conventional scale L_* given by the equation:

$$L_*^5 = \int_0^\infty f(r,t)r^4 dr \qquad (6,6)$$

where L_* like L is a certain statistically derived length characterizing the scale of the turbulence. The introduction

of this "scale" would be very convenient since from equation (6,5) there would then immediately follow:

$$\overline{v^2} \, L_*^5 = \text{constant} \qquad\qquad (6,7)$$

that is, *the product of the square of the intensity of the turbulence by the fifth power of the scale is a constant magnitude*, a very clear and simple expression of the theorem of the conservation of the disturbance moment.

It may be observed that the theorem of the conservation of the disturbance moment serves as an interesting analogy of the known fact of the conservation of the total quantity of heat during heat propagation in a fluid. This consideration is essential for the subsequent study and is briefly explained. Recalling that the Laplacian, in an n-dimensional space for a function depending only on the distance, is determined by the formula:

$$\nabla^2_{(n)} = \frac{\partial^2 F}{\partial r^2} + \frac{n-1}{r} \frac{\partial F}{\partial r}$$

it is readily seen that equation (5,9) may be interpreted as *the equation of the propagation of heat in a fluid in five-dimensional space*, the moment function $F(r,t)$ being interpreted as a *temperature* and the second term on the left representing *the convective variation of the temperature F* (its transport) expressed through the function H. With this interpretation the disturbance moment appears no other than the quantity of heat in a five-dimensional space and this quantity naturally remains constant. The foregoing analogy between the propagation of turbulent disturbances in three-dimensional space and the propagation of heat in five-dimensional space will be of use later on.

7. DECAY OF TURBULENCE - CASE OF DISTURBANCE CENTERS

OF THE "SOURCE" TYPE - LAWS OF DECAY OF

TURBULENCE OF A GIVEN INITIAL INTENSITY AND SCALE

It has been shown that the fundamental equation by Von Kármán is an indeterminate equation and for this reason the

problem of the propagation of turbulence still remains essentially undetermined. It is possible, nevertheless, to derive several very important conclusions from this equation.

Von Kármán and Howarth (reference 2) have solved the problem of the decay of the turbulence intensity behind centers of disturbance of only the point source type and leave the problem far from completely solved. Their method still leaves the exponent in the law of the decrease of the turbulence intensity undetermined, that is, only the character of the solution is given. Their computation procedure is briefly described.

In equation (5,10) r is equated to zero and since for small values of r the function h is of the order of r^3 (see (6,4)) equation (5,10) then becomes

$$\frac{d\overline{v^2}}{dt} = 2v\overline{v^2}\left(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r}\frac{\partial f}{\partial r}\right)_{r=0} = 10vf_0{}''\overline{v^2} \qquad (7,1)$$

If, according to Taylor, the function $g(r,t)$ is approximately replaced by the parabola

$$g(r,t) = g(0,t) + \frac{1}{2}g''(0,t)r^2 = 1 + \frac{1}{2}g_0{}''r^2$$

and the abscissa λ found of the intersection of this parabola with the axis r, that is, λ determined by the formula

$$\lambda^2 = -\frac{2}{g_0{}''}$$

this magnitude may be considered as an approximate characteristic of the scale of turbulence L_g strictly given by the equation:

$$L_g = \int_0^\infty g(r,t)dr$$

Passing from the function g to the function f and remembering that by (4,7)

$$f'' + 2\left(\frac{f-g}{r}\right)' = f'' + 2\,\frac{r(f'-g')-f+g}{r^2}$$

$$= f'' + 2\,\frac{f'-g'}{r} - 2\,\frac{f-g}{r^2} = 0$$

and for $r = 0$

$$f_0'' + 2(f_0'' - g_0'') - (f_0'' - g_0'') = 0$$

that is,

$$f_0'' = \frac{1}{2}\,g_0''$$

results in

$$\lambda^2 = -\frac{1}{f_0''} \qquad\qquad (7,2)$$

Equation (7,1) takes the form of the well-known Taylor equation:

$$\frac{d\overline{v^2}}{dt} = -10\nu\,\frac{\overline{v^2}}{\lambda^2} \qquad\qquad (7,3)$$

where λ or the magnitude f_0'' associated with it remain undetermined functions of time.

Furthermore, returning to equation (5,10), Von Kármán rejects the terms containing the function h and shows that this corresponds to the case of small Reynolds number of the turbulent flow (for example, number $\sqrt{\overline{v^2}}\,\lambda/\nu$). Eliminating $\overline{v^2}$ from the equation (5,10) thus simplified with the aid of (7,1), Von Kármán obtains the equation:

$$\frac{df}{dt} = 2v \left(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r} \frac{df}{dr} - 5f_0{}''f \right) \tag{7,4}$$

Seeking to obtain a particular integral of this equation in the form of a function of a single variable $\chi = \dfrac{r}{\sqrt{vt}}$ Von Kármán reduces equation (7,4) to the form

$$f'' + \left(\frac{4}{\chi} + \frac{\chi}{4} \right) f' - 5f_0{}''f = 0 \tag{7,5}$$

where the primes denote the derivatives with respect to χ and, therefore, $f_0{}''$ is a constant which Von Kármán denotes by $-\alpha$. This constant remains an undetermined constant to the end. The solution for $f(\chi)$ is given in the form:

$$f(\chi) = 2^{\frac{15}{4}} \chi^{-\frac{5}{2}} e^{-\frac{1}{16} \chi^2} M_{10\alpha - \frac{5}{4}, \frac{3}{4}} \left(\frac{\chi^2}{8} \right) \tag{7,6}$$

where M represents the known hypergeometric function.[1]

Equation (7,1) becomes

$$\frac{\overline{dv^2}}{dt} = -10v \left[\frac{d^2 f\left(\frac{r}{\sqrt{vt}} \right)}{dr^2} \right]_{r=0} \overline{v}^2 = -10 \frac{\overline{v}^2}{t} \alpha \tag{7,7}$$

It is readily integrated and gives the result

$$\frac{1}{\sqrt{\overline{v}_\sigma^2}} = \frac{1}{\sqrt{\overline{v}_0^2}} \left(\frac{t}{t_0} \right)^{5\alpha} \tag{7,8}$$

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[1] Whittaker and Watson, Course in modern analysis.

where the subscript 0 denotes the initial value. From equation (7,2)

$$ -\frac{1}{\lambda^2} = \left(\frac{\partial^2 f}{\partial r^2}\right)_{r=0} = \frac{1}{\nu t} \; f''_{\chi} (0) = -\frac{\alpha}{\nu t} $$

it also follows that

$$ \lambda^2 = \frac{\nu}{\alpha} t \tag{7,9} $$

Thus for the particular solution of $f(\chi) = f\left(\dfrac{r}{\sqrt{\nu t}}\right)$, which corresponds to the analogy of a heat source disturbance, the problem remains unsolved since α is unknown. Von Karman mentions the fact that the value $\alpha = 1/5$ corresponds to the Taylor theory.

The present author procceded in a manner differing from that of Von Kármán's and solved the problem, not only for the "point source" but also for the case of any initial disturbance.[1] In place of the correlation functions f and h consider the previously introduced moment functions F and H, that is, start not from equation (4,10) but from equation (5,9).

Following Von Kármán, the convective term is rejected and the equation

$$ \frac{\partial F}{\partial t} = 2\nu \left(\frac{\partial^2 F}{\partial r^2} + \frac{4}{r} \frac{\partial F}{\partial r}\right) \tag{7,10} $$

is solved corresponding to pure (molecular) diffusive disturbances. This equation is regarded as the equation of the propagation of heat in five-dimensional space.

Beginning with the case of a source the particular solution of equation (7,10) in the case of a source in five-

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[1] Recently the same problem of a source constituted the subject of a dissertation for a doctor's degree by M. D. Millionshikov whose method differs from the author's method. (Rep. of the Acad. of Sci., 1939.)

dimensional space is well known.[1] The solution is of the
form:

$$F(r,t) = \text{constant } \frac{e^{-\frac{r^2}{8\nu t}}}{(\sqrt{\nu t})^5} \qquad (7,11)$$

For   r = 0

$$\overline{v}^2(t) = F(0,t) = \frac{\text{constant}}{(\sqrt{\nu t})^5} \qquad (7,12)$$

Substituting the value of $v_0^2(t)$ for any $t = t_0 \neq 0$ and
and denoting it by $\overline{v}_0^2$ the equation reads

$$\frac{1}{\sqrt{\overline{v}^2}} = \frac{1}{\sqrt{\overline{v}_0^2}} \left(\frac{t}{t_0}\right)^{\frac{5}{4}} \qquad (7,13)$$

Comparing this result with the Von Kármán formula (7,8) it
is seen that the constant α introduced by him as an unknown
has a completely defined value, namely,

$$\alpha = \frac{1}{4} \qquad (7,14)$$

Finally the constant entering equations (7,11) and (7,12) can
be strictly determined. For this purpose the disturbance
moment M is found by the equation

$$M = \int_0^\infty F(r,t)r^4 dr = \text{constant} \int_0^\infty \frac{e^{-\frac{r^2}{8\nu t}}}{(\sqrt{\nu t})^5} r^4 dr$$

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[1]A. Webster, Partial Differential Equations of Mathemati-
cal Physics. (Reference is to Russian translation of book.)

The integral is readily evaluated and after simple reductions

$$\text{constant} = \frac{M}{48\sqrt{2\pi}}$$

whence

$$\bar{v}^2(t) = \frac{M}{48\sqrt{2\pi}} \frac{1}{(vt)^{\frac{5}{2}}} \qquad (7,15)$$

Making use of the obtained value of the constant by equation (6,1) gives also for the case of a point source disturbance the law of variation of the scale of turbulence L analogous to equation (7,9) for the scale according to Taylor:

$$L = \int_0^\infty \frac{F(r,t)}{\bar{v}^2} dr = \int_0^\infty e^{-\frac{r^2}{8vt}} dr = \sqrt{2\pi vt} \qquad (7,16)$$

The above equations give a complete solution of the problem of a source of given "strength" M.

If, according to Taylor, the concept of isotropy is generalized to the case of a uniform flow with average velocity U (for example, behind a screen in the working portion of a wind tunnel) and the obtained formulas applied in a Galilean system moving with velocity U then following in this manner behind the decay of the turbulence in the region of the fluid moving with velocity U, the result will be $X = U(t - t_0)$ if X is computed along the flow from a certain point corresponding to the instant, $t = t_0$. Equation (7,13) then becomes:

$$\frac{1}{\sqrt{\bar{v}^2}} = \frac{1}{\sqrt{\bar{v}_0^2}} \left(1 + \frac{X}{Ut_0}\right)^{\frac{5}{4}} \qquad (7,17)$$

As is known, Taylor gave a linear law, which is sufficiently well confirmed by experiment; while our exponent differs from unity. The reason for this probably lies in the fact that

the disturbances in the tests are not of the source type
but initial disturbances of finite magnitude and also in
the fact that pure diffusion without convection were
considered.

The problem of the decay of the intensity of the
turbulent disturbances for a given initial distribution
of the moment function satisfying only the condition of
finite disturbance reduces to the integration of equation
(7,10) for the initial condition

$$t = 0, \quad F = F_0(r) \tag{7,18}$$

Turning to the analogy with the propagation of heat in a
five-dimensional space, the general solution of the problem
is written in the form:

$$F(r,t) = \frac{1}{(2\sqrt{2\pi\nu})^5} \iiiint\limits_{-\infty}^{+\infty} F_0(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) t^{-\frac{5}{2}} e^{-\frac{\rho^2}{8\nu t}} d\alpha_1 \ldots d\alpha_5 \tag{7,19}$$

where

$$\rho^2 = (X_1 - \alpha_1)^2 + (X_2 - \alpha_2)^2 + (X_3 - \alpha_3)^2 + \ldots + (X_5 - \alpha_5)^2;$$

$$r^2 = X_1^2 + X_2^2 + \ldots X_5^2 \tag{7,20}$$

Passing to spherical coordinates the choice for the element
of volume may be an infinitely thin spherical layer of volume
equal to the product of the area of a five-dimensional sphere
of radius ρ by $d\rho$, that is,

$$\frac{8}{3} \pi^2 \rho^4 d\rho$$

Equations (7,19) then becomes

$$F(r,t) = \frac{1}{48\sqrt{2\pi}} \frac{1}{(\sqrt{vt})^5} \int_0^\infty \bar{F}_0(r,\rho,t)\rho^4 e^{-\frac{\rho^2}{8vt}} d\rho \quad (7,21)$$

where $\bar{F}_0(r,\rho,t)$ denotes the average value of the function F_0 on a sphere of radius ρ described about the point at distance r from the origin of coordinates. Passing to the limit when $r \to 0$ it is noted that on account of the assumed initial distribution of F as a function only of the distance r

$$\bar{F}_0 = F_0(\rho)$$

whence finally

$$\bar{v}^2(t) = F(0,t) = \frac{1}{48\sqrt{2\pi}} \frac{1}{(\sqrt{vt})^5} \int_0^\infty F_0(\rho)e^{-\frac{\rho^2}{8vt}}\rho^4 d\rho \quad (7,22)$$

This simple equation determines the law of decay of the turbulence intensity given by its initial distribution.

The above equation changes to equation (7,15) if, recalling the definition of source, the function $F(\rho)$ is chosen thus:

$$F_0(\rho) = 0 \quad \rho > 0$$
$$F_0(\rho) = \infty \quad \rho = 0$$

$$\int_0^\infty F_0(\rho)\rho^4 d\rho = M$$

It is of interest to note that for large t the asymptotic form of $v^2(t)$ for all initial distributions having the same disturbance agrees with the distribution for the source of the strength (moment), a property analogous to the known property of the distribution of heat.

It is seen that the initial distribution of the disturbance affects the law of decay of the turbulence. In the case of a source there existed an infinitely large initial intensity for an infinitely small initial scale and given strength. In order to evaluate, at least qualitatively, the effect of the initial scale consider the following initial distribution (stepped distribution):

$$F_0(r) = \overline{v}_0^{\,2} \qquad 0 \leq r \leq L_0 \left.\phantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right\}$$

$$F_0(r) = 0 \qquad r > L_0 \qquad\qquad (7,23)$$

In this simple case (7,22) affords

$$\overline{v}^2(t) \doteq \frac{\overline{v}_0^{\,2}}{48\sqrt{2\pi}\,(\sqrt{vt})^5} \int_0^{L_0} e^{-\frac{\rho^2}{8vt}} \rho^4\, d\rho$$

The integral is easily computed and gives[1]

$$\overline{v}^2(t) = \frac{8\overline{v}_0^{\,2}}{15\sqrt{\pi}} \left[\frac{3}{8}\sqrt{\pi}\ \mathrm{Erf}\left(\frac{L_0}{\sqrt{8vt}}\right) \right.$$

$$\left. - \frac{3}{4}\frac{L_0}{\sqrt{8vt}}\ e^{-\frac{L_0^{\,2}}{8vt}} \left(1 + \frac{1}{12}\frac{L_0^{\,2}}{vt}\right)\right] \qquad (7,24)$$

The effect of the initial scale for small values of $\dfrac{L_0}{\sqrt{vt}}$ may be taken into account by developing the function Erf into a series. After simple reductions it approximates to:

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[1] Erf here denotes the well known "error function"

$$\mathrm{Erf}\,x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\, dz$$

$$\overline{v}^2(t) = \frac{8 \, \overline{v}_0^{\,2}}{15 \sqrt{\pi}} \left(\frac{L_0}{\sqrt{8vt}} \right)^5 \left(1 + \frac{1}{28} \frac{L_0^{\,2}}{vt} \right) \qquad (7.25)$$

Noting that in the given case

$$M = \int_0^\infty F_\theta(r) r^4 \, dr = \frac{1}{5} \, \overline{v}_0^{\,2} \, L_0^{\,5}$$

equation (7.15) may, for convenience of comparing the obtained equation with the equation for the source $(L_0 = 0)$ of the same strength M, be rewritten as (7.25):

$$\overline{v}^2(t) = \frac{M}{48 \sqrt{2\pi} (\sqrt{vt})^5} \left(1 + \frac{1}{28} \frac{L_0^{\,2}}{vT} \right) \qquad (7.26)$$

The relative correction is seen to be proportional to the square of the initial scale.

A check of the correctness of all the obtained equations as well as of the fundamental theorem of the conservation of the disturbance moment on the basis of existing experimental data is, unfortunately, extremely difficult. The tests, which in the near future will be set up in the CAHI laboratories under the direction of E. M. Minsky, will serve further to develop the present concepts in this interesting field of turbulent motion.

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REFERENCES

1. Taylor, G. I.: Statistical Theory of Turbulence.
      Proc. Roy. Soc. London, ser. A, vol. 151, no. 873,
      1935, pp. 421–478; ser. A, vol. 156, no. 888, 1936,

2. von Kármán, Th.: The Fundamentals of the Statistical
      Theory of Turbulence. Jour. Aero. Sci., vol. 4,
      no. 4, 1937, pp. 131–138.

   von Kármán, Th., and Howarth, Leslie; On the Statistical
      Theory of Isotropic Turbulence. Proc. Roy. Soc.
      London, ser. A, vol. 164, no. 917, 1938, pp. 192–215.

3. Keller, L., and Friedmann, A.: Differentialgleichungen
      für die turbulente Bewegung einer kompressibelen
      Flüssigkeit. Proc. First Int. Cong. Appl. Mech.,
      1924, pp. 395–405.

4. Dryden, Hugh L., Schubauer, G. B., Mock, W. C., and
      Skramstad, H. K.: Measurements of Intensity and
      Scale of Wind–Tunnel Turbulence and Their Relation
      to the Critical Reynolds Number of Spheres. NACA
      Rep. No. 581, 1937.